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Simplified Theory for Composite Thin-Walled Beams

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The governing equations for composite thin-walled beams were derived. The theory presented here is suitable for either open-section or closed-section beams of any shape, laminate stacking sequence, and boundary conditions. Under more general assumptions than those of Vlasov, the equilibrium equations consist of seven ordinary differential equations. Further, these seven equations were reduced to four coupled ordinary differential equations, which govern the shear deformation of the middle surface. In the numerical examples, displacements of channel beams of composite laminates were calculated according to the present beam theory and compared with the finite element results and other existing theories.

Introduction

A NUMBER of theories for isotropic thin-walled beams have been developed. These theories were given by Vlasov,[1] Murray,[2] and Gjelsvik.[3] With the increasing application of thin-walled composite structures, development of theories for composite thin-walled beams has gained much attention in recent years. A Vlasov-type theory for composite thin-walled beams with open cross sections was established by Bauld and Tzeng.[4] The thin-walled beams considered were composed of a number of symmetric laminated plates. The Vlasov assumptions were adopted, i.e., the contour of the beam cross section does not deform in its plane; the shearing strain in the middle surface is negligible; each plate part in a cross section behaves as a thin plate that obeys the Kirchhoff hypothesis; and the normal stress in the contour direction is small compared with the axial stress. Since only symmetrical laminates were considered, the coupling between laminate membrane forces and laminate moments was absent.

Kobelev and Larichev[5] considered thin-walled beams with a closed section. The theory was also based on the assumption that the cross section of the beam does not deform in its own plane, and the laminate has a symmetric layup. Unlike the Vlasov theory, in their formulation the rotations of the cross section about two axes in the cross-sectional plane were independent of the beam flexural displacements. In other words, the flexural shearing strain in the middle surface was included. However, the torsional warping was not accounted for in their formulation. Vasil'eva[6] solved the free torsion problem of orthotropic composite thin-walled beams. He assumed that the axial strain and curvature are zero, and the cross section does not deform in its own plane.

Libove[7] established a simple theory for composite thin-walled beams with a single-cell closed cross section. The loads were applied only to the end cross sections. Instead of assuming undeformed cross sections, he assumed that the longitudinal strain was a linear function of the cross-sectional coordinates. Only membrane stress resultants were considered, and the shear flow was determined by the equilibrium condition. In his formulation the flexural shearing strain in the middle surface, but not torsional warping, was included.

Mansfield and Sobey[8] and Mansfield[9] developed theories for one- or two-cell beams. Although they considered some coupling effects of stiffnesses, they did not include transverse shear and cross-sectional warping in their expressions. Rehfield[10] gave a similar theory, in which the transverse shear and torsional warping, but not shell moment, were considered. The theory was applied to closed-section beams.

Bauchau[11] developed a thin-walled beam theory based on the assumptions that the cross section does not deform in its own plane; the out-of-plane cross-sectional warping was arbitrary and was expressed in terms of the so-called eigenwarpings. This theory is valid for closed, multicell beams with orthotropic material properties. However, one axis of orthotropy must be parallel to the axis of the beam, and the other axis must be parallel to the tangent of the cross-sectional contour curve. Subsequently, this theory was extended by Bauchau et al.[12] to allow for general orthotropic material properties. In practice, the eigenwarpings were solved using a finite element technique where the cross section was discretized. Usually only a few eigenwarpings were needed, e.g., torsional warping and shear deformation. Some other finite element techniques were proposed by Giavotto et al.,[13] Bauchau and Hong,[14-16] Stemple and Lee,[17,18] and Wu and Sun.[19]

Nonlinear beam kinematics were proposed by Danielson and Hodges.[20] The beam cross section was postulated to displace and rotate as a rigid body without explicit restrictions on the magnitude of the motion. Although the beam displacements require a one-dimensional and geometrically nonlinear global analysis, the deformation of the cross section can be determined on the basis of a linear theory. Detailed reviews on the composite thin-walled beam theories can be found in the papers of Hodges[21] and Friedmann.[22]

Although the existing theories can be used to analyze composite thin-walled beams, simplified theories are still needed to reduce the amount of the computation or to account for the strong coupling effects often found in composite laminates. Some of the theories just cited are too tedious and some involve many finite element techniques. They are not suitable for qualitative analyses.

A simplified theory is presented here, which adopts the same assumptions made by Wu and Sun.[19] This theory includes both torsional warping and shear deformation of the middle surface. The torsional warping is independent of rotational angle. Moreover, the shell moments are also considered. Therefore, the present theory is suitable for plates and shallow-section beams, as shown by Wu and Sun.[19] Instead of using finite element techniques, governing equations were derived here. Although the theory includes torsional warping and shear deformation, its formulation remains quite simple. The governing equations consist of seven ordinary differential equations, in which three unknowns can be expressed in terms of four other unknowns, which correspond to the shearing strain of the middle surface. Thus, only four differential equations need to be solved.

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Geometrical Relationships

Let (x, y, z) be a fixed Cartesian coordinate system, with the z axis parallel to the axis of the beam. The plane normal to the longitudinal axis z cuts the middle surface of the beam cross section with a curved line called the contour. A local contour coordinate system (n, s, z), as shown in Fig. 1, is placed on the middle surface where n and s are normal and tangential directions to the contour, respectively. In Fig. 2, point O is the initial point along the contour line coordinate s, and point P is called the pole, through which the axis parallel to the z axis is called the pole axis. Let an auxiliary Cartesian coordinate system $(\bar{n}, \bar{s})$ with axes parallel to the directions of n and s at A be placed at P. Thus, the coordinates of point A in the $(\bar{n}, \bar{s})$ coordinate system will be r and q. The coordinate system (n, s, z) can be made parallel to the system (x, y, z) by rotating through an angle $(\theta - \pi/2)$.

To derive the governing equation for a laminated composite thin-walled beam, the following assumptions are made.

Assumption 1

The contour of the thin wall does not deform in its own plane.

According to this assumption, the midsurface displacement components $\bar{u}$ (along the n direction) and $\bar{v}$ (along the s direction) at point A in the contour coordinate system can be expressed in terms of displacements U, V of the pole P in the x, y directions, respectively, and the rotation angle Φ about the pole axis, i.e.,

$$\bar{u}(s, z) = U(z)\sin\theta(s) - V(z)\cos\theta(s) - \Phi(z)q(s) \tag{1a}$$

$$\bar{v}(s, z) = U(z)\cos\theta(s) + V(z)\sin\theta(s) + \Phi(z)r(s) \tag{1b}$$

Assumption 1 is more accurate if the cross section of the thin-walled beam is stiffened by closely spaced ribs.

Assumption 2

The axial displacement $\bar{w}$ on the contour can be expressed in the following form:

$$\bar{w}(s, z) = W(z) + \xi(z)x(s) + \eta(z)y(s) + \Psi(z)\omega(s) \tag{2}$$

where W represents the average axial displacement of the beam in the z direction; x and y are the coordinates of the contour in the (x, y, z) coordinate system; and ω is the so-called sectorial coordinate or warping function, which can be determined by

$$\omega(s) = \int r(s)\, \mathrm{d}s$$

for open sections or

$$\omega(s) = \int \left[r(s) - \frac{C_s(s)}{h(s)} \right] \mathrm{d}s \tag{3}$$

for closed sections. In Eq. (3), C_s is the St. Venant shear flow (see Gjelsvik[3]), and h is wall thickness. In the classical theory of thin-walled beams, it is commonly set $\xi = -U'$, $\eta = -V'$, $\Psi = -\Phi'$. The prime denotes the derivative with respect to z. The expression given by Eq. (2) allows shear deformations in the middle surface.

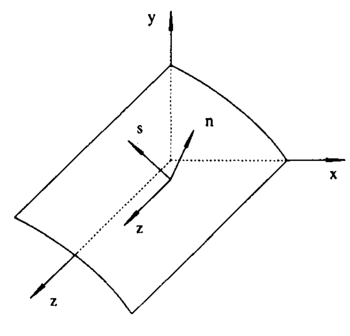

Fig. 1 Coordinate systems for thin-walled beams.

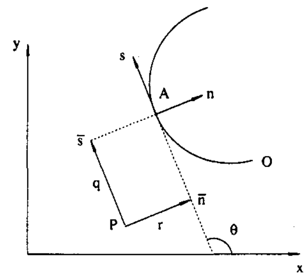

Fig. 2 Definitions of coordinates.

Several useful relations are

$$\dot{x} = \cos\theta, \qquad \dot{y} = \sin\theta, \qquad \dot{\omega} = r - \frac{C_s}{h}, \qquad \dot{q} = 1 - \frac{r}{a} \tag{4}$$

where the dot denotes the derivative with respect to s, and $a(s)$ is the radius of curvature of the contour defined to be positive when the center of curvature lies on the negative n axis.

Assumption 3

The Kirchhoff-Love assumption in classical shell theory remains valid for laminated composite thin-walled beams.

The displacements of a point off the middle surface of the wall follow assumption 3, i.e.,

$$u = \bar{u}, \qquad v = \bar{v} - \left(\frac{\partial \bar{u}}{\partial s} - \frac{\bar{v}}{a} \right)\zeta, \qquad w = \bar{w} - \frac{\partial \bar{u}}{\partial z}\zeta$$

where ζ is the thickness coordinate (in the n direction).

The strains across the wall thickness are given by

$$\epsilon_s = \frac{\epsilon_{s0} - \zeta\chi_s}{1 + \zeta/a}, \qquad \epsilon_z = \epsilon_{z0} - \zeta\chi_z$$

$$\gamma_{sz} = \frac{\mu_{zs} - \zeta\kappa_{zs}}{1 + \zeta/a} + \mu_{sz} - \zeta\kappa_{sz} \tag{5}$$

where

$$\epsilon_{z0} = \frac{\partial \bar{w}}{\partial z}, \qquad \epsilon_{s0} = \frac{\partial \bar{v}}{\partial s} + \frac{\bar{u}}{a} \tag{6a}$$

$$\mu_{sz} = \frac{\partial \bar{v}}{\partial z}, \qquad \mu_{zs} = \frac{\partial \bar{w}}{\partial s} \tag{6b}$$

$$\chi_z = \frac{\partial^2 \bar{u}}{\partial z^2}, \qquad \chi_s = \frac{\partial}{\partial s}\left(\frac{\partial \bar{u}}{\partial s} - \frac{\bar{v}}{a} \right) \tag{6c}$$

$$\kappa_{sz} = \frac{\partial}{\partial z}\left(\frac{\partial \bar{u}}{\partial s} - \frac{\bar{v}}{a} \right), \qquad \kappa_{zs} = \frac{\partial^2 \bar{u}}{\partial z \partial s} \tag{6d}$$

According to Eqs. (1), it can be shown that

$$\epsilon_s = 0 \tag{7}$$

This indicates that the cross-sectional contour is inextensible.

Governing Equations

Consider a thin-walled beam composed of thin walls of composite laminates. Locally, a state of plane stress parallel to the s-z plane is assumed. Thus, the stress-strain relations for the kth lamina are given by

$$\begin{Bmatrix} \sigma_z \\ \sigma_s \\ \tau_{zs} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \epsilon_z \\ \epsilon_s \\ \gamma_{zs} \end{Bmatrix}_k \tag{8}$$

where $\bar{Q}_{ij}$ are the reduced stiffnesses of the unidirectional fiber composite relative to the (n, s, z) coordinates (see Jones[23]). The shell forces and moments over the wall thickness are defined as

$$N_z = \int \sigma_z \left(1 + \frac{\zeta}{a}\right) d\zeta, \quad N_s = \int \sigma_s \, d\zeta \tag{9a}$$

$$N_{zs} = \int \tau_{zs} \left(1 + \frac{\zeta}{a}\right) d\zeta, \quad N_{sz} = \int \tau_{sz} \, d\zeta \tag{9b}$$

$$M_z = -\int \sigma_z \left(1 + \frac{\zeta}{a}\right) \zeta \, d\zeta, \quad M_s = -\int \sigma_s \zeta \, d\zeta \tag{9c}$$

$$M_{zs} = -\int \tau_{zs} \left(1 + \frac{\zeta}{a}\right) \zeta \, d\zeta, \quad M_{sz} = -\int \tau_{sz} \zeta \, d\zeta \tag{9d}$$

Eight constitutive equations result from Eqs. (9) using Eqs. (8). Among these equations, only six are independent and can be expressed as

$$\begin{Bmatrix} N_z \\ N_s \\ N_{sz} \\ -M_z \\ -M_s \\ -M_{zs} - M_{sz} \end{Bmatrix} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{Bmatrix} \epsilon_{z0} \\ \epsilon_{s0} \\ \mu_{sz} + \mu_{zs} \\ -\chi_z \\ -\chi_s \\ -\kappa_{sz} \end{Bmatrix} \tag{10}$$

where

$$[A] = \begin{bmatrix} Q_{11}^{(0)} + \dfrac{Q_{11}^{(1)}}{a} & Q_{12}^{(0)} & Q_{16}^{(0)} \\ Q_{12}^{(0)} & Q_{22}^{(0)} - \dfrac{Q_{22}^{(1)}}{a} & Q_{26}^{(0)} - \dfrac{Q_{26}^{(1)}}{a} \\ Q_{16}^{(0)} & Q_{26}^{(0)} - \dfrac{Q_{26}^{(1)}}{a} & Q_{66}^{(0)} - \dfrac{Q_{66}^{(1)}}{a} \end{bmatrix}$$

$$[B] = \begin{bmatrix} Q_{11}^{(1)} + \dfrac{Q_{11}^{(2)}}{a} & Q_{12}^{(1)} & 2Q_{16}^{(1)} + \dfrac{Q_{16}^{(2)}}{a} \\ Q_{12}^{(1)} & Q_{22}^{(1)} - \dfrac{Q_{22}^{(2)}}{a} & 2Q_{26}^{(1)} - \dfrac{Q_{26}^{(2)}}{a} \\ Q_{16}^{(1)} & Q_{26}^{(1)} - \dfrac{Q_{26}^{(2)}}{a} & 2Q_{66}^{(1)} - \dfrac{Q_{66}^{(2)}}{a} \end{bmatrix}$$

$$[D] = \begin{bmatrix} Q_{11}^{(2)} + \dfrac{Q_{11}^{(3)}}{a} & Q_{12}^{(2)} & 2Q_{16}^{(2)} + \dfrac{Q_{16}^{(3)}}{a} \\ Q_{12}^{(2)} & Q_{22}^{(2)} - \dfrac{Q_{22}^{(3)}}{a} & 2Q_{26}^{(2)} - \dfrac{Q_{26}^{(3)}}{a} \\ 2Q_{16}^{(2)} + \dfrac{Q_{16}^{(3)}}{a} & 2Q_{26}^{(2)} - \dfrac{Q_{26}^{(3)}}{a} & 4Q_{66}^{(2)} \end{bmatrix}$$

and

$$Q_{ij}^{(0)} = \int \bar{Q}_{ij} \, d\zeta, \qquad Q_{ij}^{(1)} = \int \bar{Q}_{ij} \zeta \, d\zeta$$

$$Q_{ij}^{(2)} = \int \bar{Q}_{ij} \zeta^2 \, d\zeta, \qquad Q_{ij}^{(3)} = \int \bar{Q}_{ij} \zeta^3 \, d\zeta$$

The constitutive equations (10) can be simplified by imposing Eq. (7). However, such a condition usually results in overestimated stiffnesses. A common practice instead is to assume $\sigma_s = 0$ in the wall. This condition leads to

$$N_s = M_s = 0 \tag{11}$$

By using these equations, ϵ_{s0} and χ_s can be eliminated and the number of constitutive equations (10) can be reduced to four, which can be expressed in the form

$$\{N\} = [S]\{\epsilon\} \tag{12}$$

where

$$\{N\} = \{N_z \quad N_{sz} \quad -M_z \quad -M_{zs} - M_{sz}\}^T$$

$$\{\epsilon\} = \{\epsilon_{z0} \quad \mu_{sz} + \mu_{zs} \quad -\chi_z \quad -\kappa_{sz}\}^T$$

The principle of virtual work states that

$$\int_0^H \int_0^L \int_{-h/2}^{h/2} (\sigma_z \delta\epsilon_z + \sigma_s \delta\epsilon_s + \tau_{zs} \delta\gamma_{zs}) \left(1 + \frac{\zeta}{a}\right) d\zeta \, ds \, dz$$
$$- \int_0^H \int_0^L (p_n \delta\bar{u} + p_s \delta\bar{v} + p_z \delta\bar{w}) \, ds \, dz$$
$$- \iint_A (F_n \delta u^* + F_s \delta v^* + F_z \delta w^*) \, dA = 0 \tag{13}$$

where H is the length of the beam; L is the length of the contour; h is the thickness of the wall; p_n, p_s, and p_z are the surface tractions along the wall in the n, s, and z directions, respectively; A are the boundary sections (i.e., end sections at $z = 0$ and H, longitudinal edge sections at $s = 0$ and L) of the beam; F_n, F_s, and F_z are the external tractions acting on the boundary sections in the n, s, and z directions, respectively; and u^*, v^*, and w^* are the displacements in the boundary sections.

Substituting Eqs. (1), (2), (4–7), and (9) into the principle of virtual work (13), we obtain the equilibrium equations:

$$\int_0^L N_{z,z} \, ds + q_0 = 0 \tag{14a}$$

$$\int_0^L (N_{z,z} x - N_{sz} \dot{x}) \, ds + q_x = 0 \tag{14b}$$

$$\int_0^L (N_{z,z} y - N_{sz} \dot{y}) \, ds + q_y = 0 \tag{14c}$$

$$\int_0^L (N_{z,z} \omega - N_{sz} \dot{\omega}) \, ds + q_\omega = 0 \tag{14d}$$

$$\int_0^L (-M_{z,zz} \sin\theta + N_{sz,z} \cos\theta) \, ds + t_x = 0 \tag{14e}$$

$$\int_0^L (M_{z,zz} \cos\theta + N_{sz,z} \sin\theta) \, ds + t_y = 0 \tag{14f}$$

$$\int_0^L (M_{z,zz} q + N_{sz,z} r - M_{sz,z} - M_{zs,z}) \, ds + t_\omega = 0 \tag{14g}$$

where

$$q_0 = \int_0^L p_z \, ds + \bar{N}_{sz} |_0^L \tag{15a}$$

$$q_x = \int_0^L p_z x \, ds + \bar{N}_{sz} x |_0^L \tag{15b}$$

$$q_y = \int_0^L p_z y \, ds + \bar{N}_{sz} y |_0^L \tag{15c}$$

$$q_\omega = \int_0^L p_z \omega \, ds + \bar{N}_{sz} \omega |_0^L \tag{15d}$$

$$t_x = \int_0^L (p_n \sin\theta + p_s \cos\theta) \, ds + [-(\bar{Q}_s + \bar{M}_{sz,z}) \sin\theta + \bar{N}_s \cos\theta] |_0^L \tag{15e}$$

$$t_y = \int_0^L (-p_n \cos\theta + p_s \sin\theta) \, ds + [(\bar{Q}_s + \bar{M}_{sz,z}) \cos\theta + \bar{N}_s \sin\theta] |_0^L \tag{15f}$$

$$t_\omega = \int_0^L (-p_n q + p_s r) \, ds + [(\bar{Q}_s + \bar{M}_{sz,z}) q + \bar{N}_s r - \bar{M}_s] |_0^L \tag{15g}$$

and

$$(\;)|_0^L = (\;)|_{s=L} - (\;)|_{s=0}$$

In Eqs. (15), $\bar{N}_{sz}$, $\bar{N}_s$, $\bar{Q}_s$, $\bar{M}_{sz}$, and $\bar{M}_s$ are external force and moment resultants acting on the two longitudinal edge boundaries ($s = 0, L$). The definitions of these external resultants are given by

$$\bar{N}_s = \int_{-h/2}^{h/2} F_s \, d\zeta, \quad \bar{N}_{sz} = \int_{-h/2}^{h/2} F_z \, d\zeta, \quad \bar{Q}_s = -\int_{-h/2}^{h/2} F_n \, d\zeta$$

$$\bar{M}_s = -\int_{-h/2}^{h/2} F_s \zeta \, d\zeta, \quad \bar{M}_{sz} = -\int_{-h/2}^{h/2} F_z \zeta \, d\zeta$$

at $s = L$. The resultants at boundary $s = 0$ are defined as they were earlier but with opposite signs. The natural boundary conditions at the ends of the beam ($z = 0, H$) are obtained from the principle of virtual work as

$$\delta W: \quad \int_0^L N_z \, ds = \int_0^L \bar{N}_z \, ds \tag{16a}$$

$$\delta \xi: \quad \int_0^L N_z x \, ds = \int_0^L \bar{N}_z x \, ds \tag{16b}$$

$$\delta \eta: \quad \int_0^L N_z y \, ds = \int_0^L \bar{N}_z y \, ds \tag{16c}$$

$$\delta \Psi: \quad \int_0^L N_z \omega \, ds = \int_0^L \bar{N}_z \omega \, ds \tag{16d}$$

$$\delta U: \quad \int_0^L (-M_{z,z} \sin\theta + N_{sz}\cos\theta) \, ds = \bar{T}_x \tag{16e}$$

$$\delta V: \quad \int_0^L (M_{z,z} \cos\theta + N_{sz}\sin\theta) \, ds = \bar{T}_y \tag{16f}$$

$$\delta \Phi: \quad \int_0^L (M_{z,z} q + N_{sz} r - M_{sz} - M_{zs}) \, ds = \bar{T}_\omega \tag{16g}$$

$$\delta U': \quad \int_0^L M_z \sin\theta \, ds = \int_0^L \bar{M}_z \sin\theta \, ds \tag{16h}$$

$$\delta V': \quad \int_0^L M_z \cos\theta \, ds = \int_0^L \bar{M}_z \cos\theta \, ds \tag{16i}$$

$$\delta \Phi': \quad \int_0^L M_z q \, ds = \int_0^L \bar{M}_z q \, ds \tag{16j}$$

in which

$$\bar{T}_x = \int_0^L (-\bar{Q}_z \sin\theta + \bar{N}_{zs}\cos\theta) \, ds + \bar{M}_{sz}\sin\theta \,|_0^L$$

$$\bar{T}_y = \int_0^L (\bar{Q}_z \cos\theta + \bar{N}_{zs}\sin\theta) \, ds - \bar{M}_{sz}\cos\theta \,|_0^L$$

$$\bar{T}_\omega = \int_0^L (\bar{Q}_z q + \bar{N}_{zs} r - \bar{M}_{zs}) \, ds - \bar{M}_{sz} q \,|_0^L$$

where

$$\bar{N}_z = \pm \int_{-h/2}^{h/2} F_z \left(1 + \frac{\zeta}{a}\right) d\zeta, \quad \bar{N}_{zs} = \pm \int_{-h/2}^{h/2} F_s \left(1 + \frac{\zeta}{a}\right) d\zeta$$

$$\bar{Q}_z = \mp \int_{-h/2}^{h/2} F_n \left(1 + \frac{\zeta}{a}\right) d\zeta, \quad \bar{M}_z = \mp \int_{-h/2}^{h/2} F_z \left(1 + \frac{\zeta}{a}\right) \zeta \, d\zeta$$

$$\bar{M}_{zs} = \mp \int_{-h/2}^{h/2} F_s \left(1 + \frac{\zeta}{a}\right) \zeta \, d\zeta$$

are the external forces and moments at $z = H$ (take the upper sign) and $z = 0$ (take the lower sign).

Further Simplification

For certain end conditions, the equilibrium equations (14) can be modified. Assume that the external forces and moments at $z = H$ are given. By integrating Eqs. (14e), (14f), and (14g) over $z = z$ to H, and then adding them to Eqs. (14b), (14c), and (14d), respectively, Eqs. (14) yield

$$\int_0^L N_{z,z} \, ds = -q_0 \tag{17a}$$

$$-\int_0^L (-M_{z,z}\dot{y} + N_{z,z} x) \, ds = q_x - T_x \tag{17b}$$

$$-\int_0^L (M_{z,z}\dot{x} + N_{z,z} y) \, ds = q_y - T_y \tag{17c}$$

$$-\int_0^L [M_{z,z} q + N_{z,z}\omega + N_{sz}(C_s/h) - M_{sz} - M_{zs}] \, ds = q_\omega - T_\omega \tag{17d}$$

$$\int_0^L (N_{z,z} x - N_{sz}\dot{x}) \, ds = -q_x \tag{17e}$$

$$\int_0^L (N_{z,z} y - N_{sz}\dot{y}) \, ds = -q_y \tag{17f}$$

$$\int_0^L (N_{z,z}\omega - N_{sz}\dot{\omega}) \, ds = -q_\omega \tag{17g}$$

where

$$T_x = \int_z^H t_x \, dz + \bar{T}_x, \quad T_y = \int_z^H t_y \, dz + \bar{T}_y, \quad T_\omega = \int_z^H t_\omega \, dz + \bar{T}_\omega$$

In the derivation of Eqs. (17), Eqs. (4), (16e), (16f), and (16g) were used. Hence these boundary conditions are satisfied. Note that Eqs. (17a-17d) are basically the same as the equilibrium equations according to the Vlasov theory.

By substituting Eqs. (1), (2), (10), and (6) into Eqs. (17), the governing equations can be expressed in the form

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{12}^T & a_{22} \end{bmatrix} \begin{Bmatrix} w_1'' \\ w_2'' \end{Bmatrix} + \begin{bmatrix} 0 & b_{12} \\ -b_{12}^T & b_{22} - b_{22}^T \end{bmatrix} \begin{Bmatrix} w_1' \\ w_2' \end{Bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & c_{22} \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{18}$$

where a prime indicates differentiation with respect to z, and

$$a_{11} = \int_0^L \begin{bmatrix} S_{11} & -(S_{11}x + S_{13}\dot{y}) & -(S_{11}y - S_{13}\dot{x}) \\ -(S_{11}x + S_{13}\dot{y}) & (S_{11}x + S_{13}\dot{y})x + (S_{13}x + S_{33}\dot{y})\dot{y} & (S_{11}y - S_{13}\dot{x})x + (S_{13}y - S_{33}\dot{x})\dot{y} \\ -(S_{11}y - S_{13}\dot{x}) & (S_{11}x + S_{13}\dot{y})y - (S_{13}x + S_{33}\dot{y})\dot{x} & (S_{11}y - S_{13}\dot{x})y - (S_{13}y - S_{33}\dot{x})\dot{x} \end{bmatrix} ds$$

$$a_{12} = \int_0^L \begin{bmatrix} -(S_{11}\omega - S_{13}q) & S_{11}x & S_{11}y & S_{11}\omega \\ (S_{11}\omega - S_{13}q)x + (S_{13}\omega - S_{33}q)\dot{y} & -(S_{11}x + S_{13}\dot{y})x & -(S_{11}x + S_{13}\dot{y})y & -(S_{11}x + S_{13}\dot{y})\omega \\ (S_{11}\omega - S_{13}q)y - (S_{13}\omega - S_{33}q)\dot{x} & -(S_{11}y - S_{13}\dot{x})x & -(S_{11}y - S_{13}\dot{x})y & -(S_{11}y - S_{13}\dot{x})\omega \end{bmatrix} ds$$

$$a_{22} = \int_0^L \begin{bmatrix} (S_{11}\omega - S_{13}q)\omega - (S_{13}\omega - S_{33}q)q & -(S_{11}\omega - S_{13}q)x & -(S_{11}\omega - S_{13}q)y & -(S_{11}\omega - S_{13}q)\omega \\ -(S_{11}\omega - S_{13}q)x & S_{11}x^2 & S_{11}xy & S_{11}x\omega \\ -(S_{11}\omega - S_{13}q)y & S_{11}xy & S_{11}y^2 & S_{11}y\omega \\ -(S_{11}\omega - S_{13}q)\omega & S_{11}x\omega & S_{11}y\omega & S_{11}\omega^2 \end{bmatrix} ds$$

$$b_{12} = \int_0^L \begin{bmatrix} S_{12}\dfrac{C_s}{h} + S_{14} & S_{12}\dot{x} & S_{12}\dot{y} & S_{12}\dot{\omega} \\ -\left(S_{12}\dfrac{C_s}{h} + S_{14}\right)x - \left(S_{23}\dfrac{C_s}{h} + S_{34}\right)\dot{y} & -(S_{12}x + S_{23}\dot{y})\dot{x} & -(S_{12}x + S_{23}\dot{y})\dot{y} & -(S_{12}x + S_{23}\dot{y})\dot{\omega} \\ -\left(S_{12}\dfrac{C_s}{h} + S_{14}\right)y + \left(S_{23}\dfrac{C_s}{h} + S_{34}\right)\dot{x} & -(S_{12}y - S_{23}\dot{x})\dot{x} & -(S_{12}y - S_{23}\dot{x})\dot{y} & -(S_{12}y - S_{23}\dot{x})\dot{\omega} \end{bmatrix} ds$$

$$
b_{22} = \int_0^L \begin{bmatrix} \left(S_{23}\dfrac{C_s}{h} + S_{34}\right)q - \left(S_{12}\dfrac{C_s}{h} + S_{14}\right)\omega & (S_{23}q - S_{12}\omega)\dot{x} & (S_{23}q - S_{12}\omega)\dot{y} & (S_{23}q - S_{12}\omega)\dot{\omega} \\ \left(S_{12}\dfrac{C_s}{h} + S_{14}\right)x & S_{12}\dot{x}x & S_{12}\dot{y}x & S_{12}\dot{\omega}x \\ \left(S_{12}\dfrac{C_s}{h} + S_{14}\right)y & S_{12}\dot{x}y & S_{12}\dot{y}y & S_{12}\dot{\omega}y \\ \left(S_{12}\dfrac{C_s}{h} + S_{14}\right)\omega & S_{12}\dot{x}\omega & S_{12}\dot{y}\omega & S_{12}\dot{\omega}\omega \end{bmatrix} \mathrm{d}s
$$

$$
c_{22} = \int_0^L \begin{bmatrix} \left(S_{22}\dfrac{C_s}{h} + 2S_{24}\right)\dfrac{C_s}{h} + S_{44} & \left(S_{22}\dfrac{C_s}{h} + S_{24}\right)\dot{x} & \left(S_{22}\dfrac{C_s}{h} + S_{24}\right)\dot{y} & \left(S_{22}\dfrac{C_s}{h} + S_{24}\right)\dot{\omega} \\ \left(S_{22}\dfrac{C_s}{h} + S_{24}\right)\dot{x} & S_{22}\dot{x}^2 & S_{22}\dot{x}\dot{y} & S_{22}\dot{x}\dot{\omega} \\ \left(S_{22}\dfrac{C_s}{h} + S_{24}\right)\dot{y} & S_{22}\dot{x}\dot{y} & S_{22}\dot{y}^2 & S_{22}\dot{y}\dot{\omega} \\ \left(S_{22}\dfrac{C_s}{h} + S_{24}\right)\dot{\omega} & S_{22}\dot{x}\dot{\omega} & S_{22}\dot{y}\dot{\omega} & S_{22}\dot{\omega}^2 \end{bmatrix} \mathrm{d}s
$$

$$
\{w_1\} = \begin{Bmatrix} W \\ U' \\ V' \end{Bmatrix}, \qquad \{w_2\} = \begin{Bmatrix} \Phi' \\ \xi + U' \\ \eta + V' \\ \Psi + \Phi' \end{Bmatrix}
$$

$$
\{f_1\} = \begin{Bmatrix} -q_0 \\ q_x - T_x \\ q_y - T_y \end{Bmatrix}, \qquad \{f_2\} = \begin{Bmatrix} q_\omega - T_\omega \\ -q_x \\ -q_y \\ -q_\omega \end{Bmatrix}
$$

The first three equations in Eqs. (18) can be solved for $\{w_1''\}$. After integrating over z to H, we have

$$
\{w_1'\} = -a_{11}^{-1}a_{12}\{w_2'\} - a_{11}^{-1}b_{12}\{w_2\} + a_{11}^{-1}\{F_1\} \tag{19}
$$

in which

$$
\{F_1\} = \begin{Bmatrix} \int_z^H q_0\,\mathrm{d}z + \int_0^L \bar{N}_z\,\mathrm{d}s \\ \int_z^H (T_x - q_x)\,\mathrm{d}z + \int_0^L \bar{M}_z\dot{y}\,\mathrm{d}s - \int_0^L \bar{N}_z x\,\mathrm{d}s \\ \int_z^H (T_y - q_y)\,\mathrm{d}z - \int_0^L \bar{M}_z\dot{x}\,\mathrm{d}s - \int_0^L \bar{N}_z y\,\mathrm{d}s \end{Bmatrix}
$$

Note that in the previous integrations boundary conditions (16a), (16b), (16c), (16h), and (16i) were used. However, conditions (16b) and (16h) were implemented together. Thus, condition (16b) is kept as an unsatisfied boundary condition. By the same token, Eq. (16c) is kept as a boundary condition to be satisfied. Using Eqs. (19) to eliminate $\{w_1\}$ from the remaining four equations in Eqs. (18), we obtain

$$
\begin{aligned}
&(a_{22} - a_{12}^T a_{11}^{-1}a_{12})\{w_2''\} + [(b_{22} - a_{12}^T a_{11}^{-1}b_{12}) \\
&\quad - (b_{22} - a_{12}^T a_{11}^{-1}b_{12})^T]\{w_2'\} - (c_{22} - b_{12}^T a_{11}^{-1}b_{12})\{w_2\} \\
&= \{f_2\} - a_{12}^T a_{11}^{-1}\{f_1\} + b_{12}^T a_{11}^{-1}\{F_1\}
\end{aligned} \tag{20}
$$

The corresponding boundary conditions at the free end $z = H$ are Eqs. (16b), (16c), (16d), and (16j). Combining Eqs. (16d) and (16j), we have

$$
-\int_0^L (M_z q + N_z\omega)\,\mathrm{d}s = -\int_0^L (\bar{M}_z q + \bar{N}_z\omega)\,\mathrm{d}s
$$

which together with Eqs. (16b), (16c), and (16d) are taken as the boundary conditions to be satisfied. These boundary conditions can be written in a simple form as

$$
a_{12}^T\{w_1'|_{z=H}\} + a_{22}\{w_2'|_{z=H}\} + b_{22}\{w_2|_{z=H}\} = \{F_2\} \tag{21}
$$

where

$$
\{F_2\} = \begin{Bmatrix} -\int_0^L (\bar{M}_z q + \bar{N}_z\omega)\,\mathrm{d}s \\ \int_0^L \bar{N}_z x\,\mathrm{d}s \\ \int_0^L \bar{N}_z y\,\mathrm{d}s \\ \int_0^L \bar{N}_z\omega\,\mathrm{d}s \end{Bmatrix}
$$

By Eqs. (19), Eqs. (21) can be rewritten as

$$
\begin{aligned}
&(a_{22} - a_{12}^T a_{11}^{-1}a_{12})\{w_2'|_{z=H}\} + (b_{22} - a_{12}^T a_{11}^{-1}b_{12})\{w_2|_{z=H}\} \\
&= \{F_2\} - a_{12}^T a_{11}^{-1}\{F_1|_{z=H}\}
\end{aligned} \tag{22}
$$

These are the equations to be used in determining the integration constants in the solution for Eqs. (20). The end conditions at $z = 0$ must also be satisfied. For a beam fixed at $z = 0$, all of the $\{w_1\} = \{w_2\} = 0$. The remaining three unknowns $\{w_1\}$ can be determined by Eqs. (19).

Examples and Discussion

The theory presented earlier considers the effects of the torsional warping and the shear deformation of the middle surface. These effects have been shown to be dominant in composite thin-walled beams by many researchers. In the present theory, the torsional warping is represented by Ψ and the shear deformation of the middle surface by

$$
\mu_{sz} + \mu_{zs} = (\xi + U')\dot{x} + (\eta + V')\dot{y} + (\Psi + \Phi')\dot{\omega} + \Phi'\,\frac{C_s}{h}
$$

The flexural shearing terms $\xi + U'$, $\eta + V'$, and $\Psi + \Phi'$ are absent in Vlasov-type theories, such as Gjelsvik[3] and Bauld and Tzeng.[4] In solid cross-sectional beam theories, the shear deformation of the middle surface is called the transverse shear deformation. From Eqs. (19), it is found that $\{w_1\}$ is statically determinate. However, $\{w_2\}$ must be obtained by solving the differential equations (20) in conjunction with the corresponding displacement boundary conditions and force boundary conditions (21).

To demonstrate the importance of shear deformation for composite thin-walled beams, solutions for laminated plates under uniaxial tension and pure shearing, respectively, are given next. The laminates considered are symmetric, so the plates are deformed in the x-z plane only. Consequently, the displacements η, Ψ, V, and Φ equal zero.

When the plate is subjected to tensile force $\bar{N}_z$ at the free end, the solution can be obtained easily as

$$W = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2}\bar{N}_z z, \qquad \xi = 0, \qquad U = \frac{-S_{12}}{S_{11}S_{22} - S_{12}^2}\bar{N}_z z$$

where S_{ij} are given by Eq. (12). Explicitly, we have

$$S_{11} = Q_{11}^{(0)} - \frac{Q_{12}^{(0)}Q_{12}^{(0)}}{Q_{22}^{(0)}}, \qquad S_{12} = Q_{16}^{(0)} - \frac{Q_{26}^{(0)}Q_{12}^{(0)}}{Q_{22}^{(0)}}$$

$$S_{22} = Q_{66}^{(0)} - \frac{Q_{26}^{(0)}Q_{26}^{(0)}}{Q_{22}^{(0)}}$$

If the plate is subjected to the shear forces $\bar{N}_{sz}$ on the boundaries $s = 0$ and L, and $\bar{N}_{zs}$ on the free end $z = H$, the displacements are obtained as

$$W = \frac{-S_{12}}{S_{11}S_{22} - S_{12}^2}\bar{N}_{sz} z, \qquad \xi = 0, \qquad U = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2}\bar{N}_{sz} z$$

under pure shearing condition $\bar{N}_{sz} = \bar{N}_{zs}$.

On the other hand, the Vlasov-type theory for composite thin-walled beams, such as Bauld and Tzeng,[4] gives the results

$$W = \frac{1}{S_{11}}\bar{N}_z z, \qquad U = 0$$

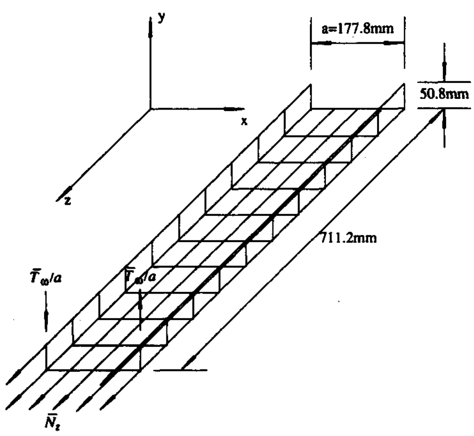

Fig. 3 Geometry of thin-walled channel.

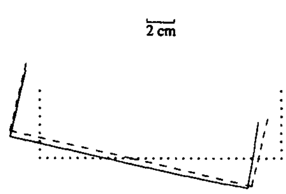

Fig. 4 Deformed contour of [45$_8$] channel under tensile load $\bar{N}_z$ = 800 KN/m (solid line: finite element; dashed line: present theory; dotted line: Bauld and Tzeng[4]).

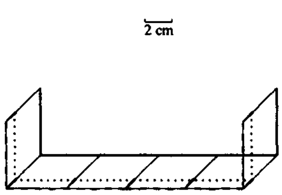

Fig. 5 Warping displacement of [45$_8$] channel under tensile load $\bar{N}_z$ = 800 KN/m (bold line: undeformed contour; solid line: finite element; dashed line: present theory; dotted line: Bauld and Tzeng[4]).

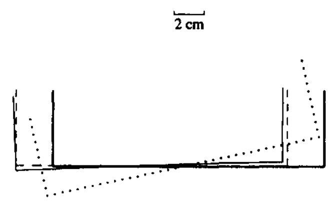

Fig. 6 Deformed contour of [45$_3$/0$_2$/45$_3$] channel under tensile load $\bar{N}_z$ = 2900 KN/m and torque $\bar{T}_\omega$ = 40 Nm (bold line: undeformed contour; solid line: finite element; dashed line: present theory; dotted line: Bauld and Tzeng[4]).

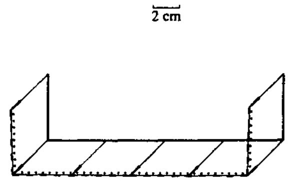

Fig. 7 Warping displacement of [45$_3$/0$_2$/45$_3$] channel under tensile load $\bar{N}_z$ = 2900 KN/m and torque $\bar{T}_\omega$ = 40 Nm (bold line: undeformed contour; solid line: finite element; dashed line: present theory; dotted line: Bauld and Tzeng[4]).

for the plate under uniaxial tension, and

$$W = 0, \qquad U = 0$$

for the plate under pure shearing.

From the preceding results, it is evident that the Vlasov theory may be inadequate for composite thin-walled beams.

The second example is a channel under uniaxial tension. The geometry of the channel is shown in Fig. 3. One end is clamped, and the other is free. A tensile force $\bar{N}_z$ = 800 kN/m is applied at the free end. The thin walls are composed of eight-ply [45]$_8$ laminate with the following material properties: E_1 = 138 GPa, E_2 = 10 GPa, ν_{12} = 0.29, G_{12} = 4.8 GPa, and ply thickness = 0.15 mm. The channel is made by folding a laminated flat plate, whose ply angle is 45 deg with respect to the z direction.

The results from the present simplified composite thin-walled beam theory are compared with shell finite element results and the results using the Bauld and Tzeng[4] theory. The finite element program used is a commercial package MARC, in which a rectangular doubly curved shell element was chosen, which has 4 nodes and 12 degrees of freedom per node. Grids used are 6 × 10 for the channel, as shown in Fig. 3. A refined mesh that doubles the number of elements along the contour was also used to check the convergence of finite element results. The deformed shape of the end cross section in its own plane is depicted in Fig. 4, and the warping displacement is shown in Fig. 5. It is found that uniaxial tension causes significant bending and torsion. The theories that neglect the transverse shear, such as Bauld and Tzeng,[4] do not yield any displacements in the cross-sectional plane, as shown by the dotted line in Fig. 4. The results from the present theory and the finite element method agree quite well. These extension-twist and extension-bending coupling phenomena can be used to suppress or to enhance some displacements.

The third example is a channel composed of the [45$_3$/0$_2$/45$_3$] laminate. The free end is subjected to both a tensile load $\bar{N}_z$ = 2900 kN/m and a torque $\bar{T}_\omega$ = 40 Nm that is represented by two point loads in the finite element model as shown in Fig. 3. From Fig. 6 we can see that the rotation is suppressed despite of the torque. However, the results using the Bauld and Tzeng[4] theory yielded a significant rotation.

Summary

A simplified theory for composite thin-walled beams has been developed. The formulation of this theory is simple, and the solution can be readily obtained by solving four equations. It is suitable for either open-section or closed-section beams of any shape, laminate stacking sequence, and boundary conditions. In spite of its simplicity, the effects of torsional warping and transverse shearing deformation, two dominant modes of deformation in composite thin-walled beams, are included. Furthermore, since the shell moments are included, the present theory is capable of analyzing flat plates as a special case of thin-walled beams for which some theories[7-12] are inadequate.

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